BOREL MEASURES WITH A DENSITY ON A COMPACT SEMI-ALGEBRAIC SET

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Abstract. Let $\mathbf{K} \subset \mathbb{R}^n$ be a compact basic semi-algebraic set. We provide a necessary and sufficient condition (with no *à priori* bounding parameter) for a real sequence $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, to have a finite representing Borel measure absolutely continuous w.r.t. the Lebesgue measure on $\mathbf{K}$, and with a density in $\cap_{p=1}^{\infty} L_p(\mathbf{K})$. With an additional condition involving a bounding parameter, the condition is necessary and sufficient for existence of a density in $L_\infty(\mathbf{K})$. Moreover, nonexistence of such a density can be detected by solving finitely many of a hierarchy of semidefinite programs. In particular, if the semidefinite program at step d of the hierarchy has no solution then the sequence cannot have a representing measure on $\mathbf{K}$ with a density in $L_p(\mathbf{K})$ for any $p \geq 2d$.

1. Introduction

The famous Markov moment problem (also called the L-problem of moments) is concerned with characterizing real sequences (s_n), $n \in \mathbb{N}$, which are moment of a Borel probability measure μ on $[0,1]$ with a bounded density with respect to (w.r.t.) the Lebesgue measure. It was posed by Markov and later solved by Hausdorff with the following necessary and sufficient condition:

$$1 = s_0 \quad \text{and} \quad 0 \leq s_{nj} \leq c/(n+1), \qquad \forall\, n, j \in \mathbb{N}, \tag{1.1}$$

for some $c > 0$, where $s_{nj} := (-1)^{n-j}\binom{n}{j}\Delta^{n-j}s_j$, and Δ is the forward operator $s_n \mapsto \Delta s_n = s_{n+1} - s_n$. Similarly, with $p > 1$, if in (1.1) one replaces the condition "$s_{nj} \leq c/(n+1)$ for all $n, j \in \mathbb{N}$", with the condition

$$\sup_n \left(\frac{1}{n+1} \sum_{j=0}^{n} ((n+1)\, s_{nj})^p \right)^{1/p} < c, \tag{1.2}$$

then one obtains a characterization of real sequences having a representing Borel measure with a density in $L_p([0,1])$ with p-norm bounded by c. For an illuminating discussion with historical remarks the reader is referred to Diaconis and Freedman [3] where the authors also make a connection with De Finetti's theorem on exchangeable 0-1 valued random variables. In addition, Putinar [9, 10] has provided a characterization of extremal solutions of the two-dimensional L-problem of moments.

An alternative *if and only if* characterization is via Linear Matrix Inequalities (LMIs) involving the moment and localizing matrices associated with the real sequence and the Lebesgue measure λ on $[0,1]$, respectively. It states that the moment

1991 *Mathematics Subject Classification.* 44A60 47A57 90C22.

Key words and phrases. L-moment problem; Borel measures with a density; semidefinite programming.

and localizing matrices of the sequence must be dominated by that of λ (scaled with some factor $c > 0$). This latter characterization carries over arbitrary compact basic semi-algebraic sets of $\mathbb{R}^n$ and a similar characterization also holds for measures on non-compact sets, satisfying an additional Carleman type condition; see e.g. Lasserre [5, p. 67–69].

A common feature of the above characterizations is the *à priori* unknown scalar $c > 0$ which is precisely the required bound on the density. So for instance, if the conditions (1.1) fail one does not know whether it is because the sequence has no representing measure μ with a density in $L_\infty([0,1])$ or because c is not large enough. So it is more appropriate to state that (1.1) are necessary and sufficient conditions for μ to have a density $f \in L_\infty([0,1])$ *with* $\|f\|_\infty \leq c$. And similarly for (1.2) for $f \in L_p([0,1])$ *with* $\|f\|_p \leq c$.

Contribution: Consider a compact basic semi-algebraic set $\mathbf{K} \subset \mathbb{R}^n$ of the form

$$\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n \,:\, g_j(\mathbf{x}) \geq 0,\ j = 1, \ldots, m\}, \tag{1.3}$$

for some polynomials $g_j \in \mathbb{R}[\mathbf{x}]$, $j = 1, \ldots, m$. For every, $p \in \mathbb{N}$, denote by $L_p(\mathbf{K})$ the Lebesgue space of functions such that $\int_{\mathbf{K}} |f|^p \lambda(d\mathbf{x}) < \infty$. We then provide a set of conditions (S) with no *à priori* bound c, and such that:

- A real sequence $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, has a finite representing Borel measure with a density in $\cap_{p=1}^\infty L_p(\mathbf{K})$, if and only if (S) is satisfied. In particular, if (S) is violated we obtain a condition (with no à priori bounding parameter c) for non existence of a density in $\cap_p L_p(\mathbf{K})$ (hence no density in $L_\infty(\mathbf{K})$ either).

- A real sequence $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, has a finite representing Borel measure with a density in $L_\infty(\mathbf{K})$ if and only if (S) and an additional condition (involving an *à priori* bound $c > 0$), are satisfied.

In addition, the conditions (S) consist of a hierarchy of Linear Matrix Inequalities (LMIs) and so can be tested numerically via available semidefinite programming softwares. In particular, if a finite Borel measure μ does not have a density in $\cap_p L_p(\mathbf{K})$ (and so no density in $L_\infty(\mathbf{K})$ either), it can be detected by solving *finitely many* semidefinite programs in the hierarchy until one has no feasible solution. That is, it can be detected from finitely many of its moments. This is illustrated on a simple example.

Conversely, if the semidefinite program at step d of the hierarchy has no solution, then one may conclude that the real sequence $\mathbf{y}$ cannot have a representing measure on $\mathbf{K}$ with a density in $L_p(\mathbf{K})$, for any $p \geq 2d$.

So a distinguishing feature of our result is the absence of an *à priori* bound c in the set of condition (S) to test whether $\mathbf{y}$ has a density in $\cap_{p=1}^\infty L_p(\mathbf{K})$. Crucial for our result is a representation of polynomials that are positive on $\mathbf{K} \times \mathbb{R}$, by Powers [8]; see also Marshall [6, 7].

2. Main result

2.1. **Notation, definitions and preliminary results.** Let $\mathbb{R}[\mathbf{x}, t]$ (resp. $\mathbb{R}[\mathbf{x}, t]_d$) denote the ring of real polynomials in the variables $\mathbf{x} = (x_1, \ldots, x_n, t)$ (resp. polynomials of degree at most d), whereas $\Sigma[\mathbf{x}, t]$ (resp. $\Sigma[\mathbf{x}, t]_d$) denotes its subset of sums of squares (s.o.s.) polynomials (resp. of s.o.s. of degree at most $2d$). For every $\alpha \in \mathbb{N}^n$ the notation $\mathbf{x}^\alpha$ stands for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and for every

$d\in\mathbb{N}$, let $\mathbb{N}^{n+1}_d:=\{\beta\in\mathbb{N}^{n+1}:\sum_j\beta_j\leq d\}$ whose cardinal is $s(d)=\binom{n+d+1}{d}$. A polynomial $f\in\mathbb{R}[\mathbf{x},t]$ is written

$$(\mathbf{x},t)\mapsto f(\mathbf{x},t)\,=\,\sum_{(\alpha,k)\in\mathbb{N}^n\times\mathbb{N}} f_{\alpha k}\,\mathbf{x}^\alpha\,t^k,$$

and f can be identified with its vector of coefficients $\mathbf{f}=(f_{\alpha k})$ in the canonical basis $(\mathbf{x}^\alpha,t^k)$, $(\alpha,k)\in\mathbb{N}^n\times\mathbb{N}$, of $\mathbb{R}[\mathbf{x},t]$. But we can also write f as

$$(\mathbf{x},t)\mapsto f(\mathbf{x},t)\,=\,\sum_{k\in\mathbb{N}} f_k(\mathbf{x})\,t^k, \tag{2.1}$$

for finitely many polynomials $f_k\in\mathbb{R}[\mathbf{x}]$.

A real sequence $\mathbf{z}=(z_{\alpha k})$, $(\alpha,k)\in\mathbb{N}^n\times\mathbb{N}$, has a *representing measure* if there exists some finite Borel measure ν on $\mathbb{R}^n\times\mathbb{R}$ such that

$$z_{\alpha k}\,=\,\int_{\mathbb{R}^{n+1}}\mathbf{x}^\alpha\,t^k\,d\nu(\mathbf{x},t),\qquad\forall\,(\alpha,k)\in\mathbb{N}^n\times\mathbb{N}.$$

Given a real sequence $\mathbf{z}=(z_{\alpha k})$ define the linear functional $L_{\mathbf{y}}:\mathbb{R}[\mathbf{x},t]\to\mathbb{R}$ by:

$$f\;(=\sum_{\alpha,k} f_{\alpha k}\,\mathbf{x}^\alpha\,t^k)\quad\mapsto L_{\mathbf{z}}(f)\,=\,\sum_{\alpha,k} f_{\alpha k}\,y_{\alpha k},\qquad f\in\mathbb{R}[\mathbf{x},t].$$

Moment matrix. The *moment* matrix associated with a sequence $\mathbf{z}=(z_{\alpha k})$, $(\alpha,k)\in\mathbb{N}^n\times\mathbb{N}$, is the real symmetric matrix $\mathbf{M}_d(\mathbf{z})$ with rows and columns indexed by $\mathbb{N}^{n+1}_d$, and whose entry (α,β) is just $z_{\alpha+\beta}$, for every $\alpha,\beta\in\mathbb{N}^{n+1}_d$. Alternatively, let $\mathbf{v}_d((\mathbf{x},t))\in\mathbb{R}^{s(d)}$ be the vector $((\mathbf{x},t)^\alpha)$, $\alpha\in\mathbb{N}^{n+1}_d$, and define the matrices $(\mathbf{B}_\alpha)\subset\mathcal{S}^{s(d)}$ by

$$\mathbf{v}_d((\mathbf{x},t))\,\mathbf{v}_d((\mathbf{x},t))^T\,=\,\sum_{\alpha\in\mathbb{N}^{n+1}_{2d}}\mathbf{B}_\alpha\,(\mathbf{x},t)^\alpha,\qquad\forall(\mathbf{x},t)\in\mathbb{R}^{n+1}. \tag{2.2}$$

Then $\mathbf{M}_d(\mathbf{z})=\sum_{\alpha\in\mathbb{N}^{n+1}_{2d}} z_\alpha\,\mathbf{B}_\alpha$.

If $\mathbf{z}$ has a representing measure ν then $\mathbf{M}_d(\mathbf{z})\succeq 0$ because

$$\langle\mathbf{f},\mathbf{M}_d(\mathbf{z})\mathbf{f}\rangle\,=\,\int f^2\,d\nu\,\geq 0,\qquad\forall\,\mathbf{f}\,\in\mathbb{R}^{s(d)}.$$

Localizing matrix. With $\mathbf{z}$ as above and $g\in\mathbb{R}[\mathbf{x},t]$ (with $g(\mathbf{x},t)=\sum_\gamma g_\gamma(\mathbf{x},t)^\gamma$), the *localizing* matrix associated with $\mathbf{z}$ and g is the real symmetric matrix $\mathbf{M}_d(g\,\mathbf{z})$ with rows and columns indexed by $\mathbb{N}^{n+1}_d$, and whose entry (α,β) is just $\sum_\gamma g_\gamma z_{\alpha+\beta+\gamma}$, for every $\alpha,\beta\in\mathbb{N}^{n+1}_d$. Alternatively, let $\mathbf{C}_\alpha\in\mathcal{S}^{s(d)}$ be defined by:

$$g(\mathbf{x},t)\,\mathbf{v}_d(\mathbf{x},t)\,\mathbf{v}_d(\mathbf{x},t)^T\,=\,\sum_{\alpha\in\mathbb{N}^{n+1}_{2d+\deg g}}\mathbf{C}_\alpha\,(\mathbf{x},t)^\alpha,\qquad\forall(\mathbf{x},t)\in\mathbb{R}^{n+1}. \tag{2.3}$$

Then $\mathbf{M}_d(g\,\mathbf{z})=\sum_{\alpha\in\mathbb{N}^{n+1}_{2d+\deg g}} z_\alpha\,\mathbf{C}_\alpha$.

If $\mathbf{z}$ has a representing measure ν whose support is contained in the set $\{(\mathbf{x},t):g(\mathbf{x},t)\geq 0\}$ then $\mathbf{M}_d(g\,\mathbf{z})\succeq 0$ because

$$\langle\mathbf{f},\mathbf{M}_d(g\,\mathbf{y})\mathbf{f}\rangle\,=\,\int f^2\,g\,d\nu\,\geq 0,\qquad\forall\,\mathbf{f}\,\in\mathbb{R}^{s(d)}.$$

With $\mathbf{K}$ as in (1.3), and for every $j=0,1,\ldots,m$, let $v_j:=\lceil(\deg g_j)/2\rceil$.

Definition 2.1. *With* $\mathbf{K}$ *as in (1.3) let* $P(g) \subset \mathbb{R}[\mathbf{x}, t]$ *be the convex cone:*

$$P(g) = \left\{ \sum_{\beta \in \{0,1\}^m} \psi_\beta(\mathbf{x}, t)\, g_1(\mathbf{x})^{\beta_1} \cdots g_m(\mathbf{x})^{\beta_m} \;:\; \psi_\beta \in \Sigma[\mathbf{x}, t] \right\}. \tag{2.4}$$

The convex cone $P(g)$ is called a *preordering* associated with the g_j's.

Proposition 2.2. *Let* $\mathbf{K}$ *be as in (1.3). A polynomial* $f \in \mathbb{R}[\mathbf{x}, t]$ *is nonnegative on* $\mathbf{K} \times \mathbb{R}$ *only if* f *can be written as*

$$(\mathbf{x}, t) \mapsto \quad f(\mathbf{x}, t) = \sum_{k=0}^{2d} f_k(\mathbf{x})\, t^k, \tag{2.5}$$

for some $d \in \mathbb{N}$ *and where* $f_{2d} \geq 0$ *on* $\mathbf{K}$*.*

Proof. Suppose that the highest degree in t is $2d+1$ for some $d \in \mathbb{N}$. Then $f_{2d+1} \neq 0$ and so by fixing an arbitrary $\mathbf{x}_0 \in \mathbf{K}$, the univariate polynomial $t \mapsto f(\mathbf{x}_0, t)$ can be made negative, in contradiction with $f \geq 0$ on $\mathbf{K} \times \mathbb{R}$. Hence the highest degree in t is even, say $2d$. But then of course, for obvious reasons $f_{2d} \geq 0$ on $\mathbf{K}$. □

We have the following important preliminary result.

Theorem 2.3 ([6, 8])**.** *Let* $\mathbf{K}$ *as in (1.3) be compact and let* $f \in \mathbb{R}[\mathbf{x}, t]$ *be of the form* $f(\mathbf{x}, t) = \sum_{k=0}^{2d} f_k(\mathbf{x}) t^k$ *for some polynomials* $(f_k) \subset \mathbb{R}[\mathbf{x}]$*, and with* $f_{2d} > 0$ *on* $\mathbf{K}$*. Then* $f \in P(g)$ *if* $f > 0$ *on* $\mathbf{K} \times \mathbb{R}$*.*

And so we can derive a version of the $\mathbf{K} \times \mathbb{R}$-moment problem where for each $\beta \in \mathbb{N}^m$, the notation g^β stands for the polynomial $g_1^{\beta_1} \cdots g_m^{\beta_m}$.

Corollary 2.4. *Let* $\mathbf{K}$ *as in (1.3) be compact. A real sequence* $\mathbf{z} = (z_{\alpha k})$*,* $(\alpha, k) \in \mathbb{N}^n \times \mathbb{N}$*, has a representing measure on* $\mathbf{K} \times \mathbb{R}$ *if and only if*

$$\mathbf{M}_d(\mathbf{z}) \succeq 0; \quad \mathbf{M}_d(g^\beta\, \mathbf{z}) \succeq 0, \qquad \beta \in \{0,1\}^m, \tag{2.6}$$

for every $d \in \mathbb{N}$*.*

Proof. The *only if* part is straightforward from the definition of the moment and localizing matrix $\mathbf{M}_d(\mathbf{z})$ and $\mathbf{M}_d(g^\beta\, \mathbf{z})$, respectively.

The *if part.* Suppose that (2.6) holds true, and let $f \in \mathbb{R}[\mathbf{x}, t]$ be nonnegative on the closed set $\mathbf{K} \times \mathbb{R}$. Hence by Proposition 2.2, f has the decomposition (2.5) for some integer $d \neq 0$. For every $\epsilon > 0$, the polynomial $(\mathbf{x}, t) \mapsto f_\epsilon(\mathbf{x}, t) := f(\mathbf{x}, t) + \epsilon(1 + t^{2d})$ has the decomposition

$$f_\epsilon(\mathbf{x}, t) = \sum_{k=0}^{2d} f_{\epsilon k}(\mathbf{x})\, t^k,$$

with $f_{\epsilon 0} = f_0 + \epsilon$ and $f_{\epsilon 2d}(\mathbf{x}) = f_{2d}(\mathbf{x}) + \epsilon$. Therefore, f_ϵ is strictly positive on $\mathbf{K} \times \mathbb{R}$, and $f_{\epsilon 2d} > 0$ on $\mathbf{K}$. By Theorem 2.3, $f_\epsilon \in Q(g)$, i.e.,

$$f_\epsilon(\mathbf{x}, t) = \sum_{\beta \in \{0,1\}^m} \psi_\beta(\mathbf{x}, t)\, g(\mathbf{x})^\beta,$$

for some SOS polynomials $(\psi_\beta)\subset\Sigma[\mathbf{x},t]$. Next, let $\mathbf{z}$ satisfy (2.6). Then

$$\begin{aligned}L_{\mathbf{z}}(f)+\epsilon\,L_{\mathbf{y}}(1+t^{2d}) &= L_{\mathbf{z}}(f_\epsilon)\\ &= \sum_{\beta\in\{0,1\}^m} L_{\mathbf{z}}(\psi_\beta\,g^\beta)\;\geq\;0\end{aligned}$$

where the last inequality follows from

$$\mathbf{M}_d(g^\beta\,\mathbf{z})\succeq 0\;\Leftrightarrow\;L_{\mathbf{z}}(h^2\,g^\beta)\;\geq\;0,\quad\forall h\in\mathbb{R}[\mathbf{x},t]_d,$$

for every $\beta\in\{0,1\}^m$. But since $L_{\mathbf{z}}(1+t^{2d})\geq 0$ and $\epsilon>0$ was arbitrary, one may conclude that $L_{\mathbf{z}}(f)\geq 0$ for every $f\in\mathbb{R}[\mathbf{x},t]$ which is nonnegative on $\mathbf{K}\times\mathbb{R}$. Hence by the Riesz-Haviland theorem (see e.g. [5, Theorem 3.1, p. 53]), $\mathbf{z}$ has a representing measure on $\mathbf{K}\times\mathbb{R}$. □

2.2. **Main result.** Let $L_\infty(\mathbf{K})$ be the Lebesgue space of integrable functions on $\mathbf{K}$ (with respect to the Lebesgue measure λ on $\mathbf{K}$, scaled to a probability measure) and essentially bounded on $\mathbf{K}$. And with $1\leq p<\infty$, let $L_p(\mathbf{K})$ be the Lebesgue space of integrable functions f on $\mathbf{K}$ such that $\int_{\mathcal{K}}|f|^p\lambda(d\mathbf{x})<\infty$. A Borel measure μ absolutely continuous w.r.t. λ is denoted $\mu\ll\lambda$.

Theorem 2.5. *Let $\mathbf{K}\subset\mathbb{R}^n$ be as in (1.3) and let $\mathbf{y}=(\mathbf{y}_\alpha)$, $\alpha\in\mathbb{N}^n$, be a real sequence with $y_0=1$. Then the following two propositions (i) and (ii) are equivalent:*

(i) $\mathbf{y}$ has a representing Borel probability measure $\mu\ll\lambda$ on $\mathbf{K}$, with a density in $\cap_{p=1}^\infty L_p(\mathbf{K})$.

(ii) $\mathbf{M}_d(\mathbf{y})\succeq 0$ and $\mathbf{M}_d(g^\beta\,\mathbf{y})\succeq 0$ for all $\beta\in\{0,1\}^m$ and all $d\in\mathbb{N}$. In addition, there exists a sequence $\mathbf{z}=(z_{\alpha k})$, $(\alpha,k)\in\mathbb{N}^n\times\mathbb{N}$, such that (2.6) holds, and

$$z_{\alpha 0}\;=\;\int_{\mathbf{K}}\mathbf{x}^\alpha\,\lambda(d\mathbf{x})\,;\quad z_{\alpha 1}\;=\;y_\alpha,\quad\forall\alpha\in\mathbb{N}^n.\tag{2.7}$$

Moreover, if in (2.7) one includes the additional condition $\sup_k z_{0k}<\infty$, then (ii) is necessary and sufficient for $\mathbf{y}$ to have a representing Borel probability measure $\mu\ll\lambda$ on $\mathbf{K}$, with a density in $L_\infty(\mathbf{K})$.

Proof. The (i) ⇒ (ii) implication. As $\mathbf{y}$ has a representing Borel probability measure μ on $\mathbf{K}$ with a density $f\in L_p(\mathbf{K})$ for every $p=1,2,\ldots$, one may write

$$\mu(A)\;=\;\int_A f(\mathbf{x})\,\lambda(d\mathbf{x}),\qquad\forall A\;\in\;\mathcal{B}(\mathbb{R}^n).$$

Define the stochastic kernel $\varphi(B|\mathbf{x})$, $B\in\mathcal{B}(\mathbb{R})$, $\mathbf{x}\in\mathbf{K}$, where for almost all $\mathbf{x}\in\mathbf{K}$, $\varphi(\cdot\,|\,\mathbf{x})$ is the Dirac measure at the point $f(\mathbf{x})$. Next, let ν be the finite Borel measure on $\mathbf{K}\times\mathbb{R}$ defined by

$$\nu(A\times B)\;:=\;\int_A\varphi(B|\mathbf{x})\,\lambda(d\mathbf{x}),\quad\forall A\in\mathcal{B}(\mathbb{R}^n),\;B\in\mathcal{B}(\mathbb{R}).\tag{2.8}$$

Let $\mathbf{z}=(y_{\alpha k})$, $(\alpha,k)\in\mathbb{N}^n\times\mathbb{N}$, be the sequence of moments of ν.

$$\begin{aligned}z_{\alpha k} &= \int_{\mathbb{R}}\mathbf{x}^\alpha\,t^k\,d\nu(\mathbf{x},t)\;=\;\int_{\mathbf{K}}\mathbf{x}^\alpha\left(\int_{\mathbb{R}}t^k\varphi(dt\,|\,\mathbf{x})\right)\lambda(d\mathbf{x}),\\ &= \int_{\mathbf{K}}\mathbf{x}^\alpha\,f(\mathbf{x})^k\,\lambda(d\mathbf{x})\quad\text{(well defined as } f\in L_p(\mathbf{K})\text{ for all } p).\end{aligned}\tag{2.9}$$

In particular, for every $\alpha\in\mathbb{N}^n$,

$$z_{\alpha 0}\,=\,\int_{\mathbf{K}}\mathbf{x}^\alpha\;\lambda(d\mathbf{x})\,;\quad z_{\alpha 1}\,=\,\int_{\mathbf{K}}\mathbf{x}^\alpha\,f(\mathbf{x})\,\lambda(d\mathbf{x})\,=\,\int_{\mathbf{K}}\mathbf{x}^\alpha\,d\mu\,=\,y_\alpha.$$

Moreover, as ν is supported on $\mathbf{K}\times\mathbb{R}$ then $\mathbf{M}_d(\mathbf{y})\succeq 0$ and $\mathbf{M}_d(g^\beta\,\mathbf{y})\succeq 0$ for all d and all $\beta\in\{0,1\}^m$. Hence (2.6)-(2.7) hold.

The (ii) $\Rightarrow$ (i) implication. Let $\mathbf{z}=(z_{\alpha k})$ be such that (2.6)-(2.7) hold. By Corollary (2.4), $\mathbf{z}$ has a representing Borel probability measure ν on $\mathbf{K}\times\mathbb{R}$. One may disintegrate ν in the form

$$\nu(A\times B)\,=\,\int_{A\cap\mathbf{K}}\varphi(B\,|\,\mathbf{x})\,\psi(d\mathbf{x}),\qquad B\in\mathcal{B}(\mathbb{R}),\;A\in\mathcal{B}(\mathbb{R}^n),$$

for some stochastic kernel $\varphi(\cdot\,|\,\mathbf{x})$, and where ψ is the marginal (probability measure) of ν on $\mathbf{K}$. From (2.7) we deduce that

$$\int_{\mathbf{K}}\mathbf{x}^\alpha\,\psi(d\mathbf{x})\,=\,z_{\alpha 0}\,=\,\int_{\mathbf{K}}\mathbf{x}^\alpha\,\lambda(d\mathbf{x}),\qquad\forall\alpha\in\mathbb{N}^n,$$

which, as $\mathbf{K}$ is compact, implies that $\psi=\lambda$. In addition, still from (2.7),

$$\begin{aligned} z_{\alpha 1}\,=\,\int_{\mathbf{K}}\mathbf{x}^\alpha\,t\,d\nu(\mathbf{x},t) &= \int_{\mathbf{K}}\mathbf{x}^\alpha\underbrace{\left(\int_{\mathbb{R}}t\,\varphi(dt|\mathbf{x})\right)}_{f(\mathbf{x})}\lambda(d\mathbf{x})\qquad\forall\alpha\in\mathbb{N}^n\\ &= \int_{\mathbf{K}}\mathbf{x}^\alpha\,\underbrace{f(\mathbf{x})\,\lambda(d\mathbf{x})}_{d\theta(\mathbf{x})}\qquad\forall\alpha\in\mathbb{N}^n, \end{aligned}\tag{2.10}$$

where $f:\mathbf{K}\to\mathbb{R}$ is the measurable function $\mathbf{x}\mapsto\int_{\mathbb{R}}t\,\varphi(dt|\mathbf{x})$, and θ is the signed Borel measure $\theta(B):=\int_{\mathbf{K}\cap B}f(\mathbf{x})\,\lambda(d\mathbf{x})$, for all $B\in\mathcal{B}(\mathbb{R}^n)$.

But as $\mathbf{K}$ is compact, by Schmüdgen's Positivstellensatz [11], the conditions

$$\mathbf{M}_d(\mathbf{y})\,\succeq 0,\quad\mathbf{M}_d(g^\beta\,\mathbf{y})\,\succeq\,0,\quad\beta\in\{0,1\}^m,\quad\forall d\in\mathbb{N},$$

imply that $\mathbf{y}$ has a finite representing Borel measure μ on $\mathbf{K}$. And so as $z_{\alpha 1}=y_\alpha$ for all $\alpha\in\mathbb{N}^n$, and measures on compact sets are moment determinate, one may conclude that $d\mu=d\theta=f\,d\lambda$, that is, $\mu\ll\lambda$ on $\mathbf{K}$. Next, observe that for every $p\in\mathbb{N}$,

$$\begin{aligned} z_{0p}\,=\,\int_{\mathbf{K}}t^p d\nu(\mathbf{x},t) &= \int_{\mathbf{K}}\underbrace{\left(\int_{\mathbb{R}}t^p\,\varphi(dt|\mathbf{x})\right)}_{f(\mathbf{x})^p}\lambda(d\mathbf{x})\qquad\forall\alpha\in\mathbb{N}^n\\ &= \int_{\mathbf{K}}f(\mathbf{x})^p\,\lambda(d\mathbf{x})\qquad\forall\alpha\in\mathbb{N}^n, \end{aligned}$$

and so $f\in L_p(\mathbf{K})$ for all $p\geq 1$.

Finally consider (2.7) with the additional condition $\sup_p z_{0p}<\infty$. Then in the above proof of (i) $\Rightarrow$ (ii) and since now $\mathbf{y}$ has a *finite* representing Borel measure with a density $f\in L_\infty(\mathbf{K})$, one has $\lim_{p\to\infty}\|f\|_p=\|f\|_\infty$ because $\mathbf{K}$ is compact; see e.g. Ash [2, problem 9, p. 91]. And therefore since $z_{0p}=\int_{\mathbf{K}}f(\mathbf{x})^p\lambda(d\mathbf{x})$, we obtain $\sup_p z_{0p}<\infty$.

Similarly, in the above proof of (ii) $\Rightarrow$ (i), $\sup_p z_{0p}<\infty$ implies $\sup_p\int_{\mathbf{K}}f(\mathbf{x})^p\lambda(d\mathbf{x})=\sup_p\|f\|_p<\infty$. But this implies that $f\in L_\infty(\mathbf{K})$ since $\mathbf{K}$ is compact. $\square$

Computational procedure. Let $\gamma = (\gamma_\alpha)$, $\alpha \in \mathbb{N}^n$, the moment of the Lebesgue measure on $\mathbf{K}$, scaled to make it a probability measure. In fact, the (scaled) Lebesgue measure on any box that contains $\mathbf{K}$ is fine.

Let $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, be a real given sequence, and with $\mathbf{K}$ as in (1.3) let $v_j := \lceil (\deg g_j)/2 \rceil$, $j = 1, \ldots, m$. To check the conditions in Theorem 2.5(ii), one solves the hierarchy of optimization problems, parametrized by $d \in \mathbb{N}$.

$$
\begin{array}{rll}
\rho_d = \displaystyle\min_{\mathbf{z}} & \mathrm{trace}(\mathbf{M}_d(\mathbf{z})) & \\
\text{s.t.} & \mathbf{M}_d(\mathbf{z}) \succeq 0 & \\
& \mathbf{M}_{d-v_j}(g^\beta\,\mathbf{z}) \succeq 0, \quad \beta \in \{0,1\}^m & \\
& z_{\alpha 0} = \gamma_\alpha, \quad \alpha \in \mathbb{N}^n_{2d} & \\
& z_{\alpha 1} = y_\alpha, \quad (\alpha, 1) \in \mathbb{N}^{n+1}_{2d}. &
\end{array} \tag{2.11}
$$

Each problem (2.11) is a semidefinite program[1]. Moreover, if (2.11) has a feasible solution then it has an optimal solution. This is because as one minimizes the trace of $\mathbf{M}_d(\mathbf{z})$, the feasible set is bounded and closed, hence compact.

In (2.11) one may also include the additional constraints $z_{0k} < c$, $k \leq 2d$, for some fixed $c > 0$. Then by Theorem 2.5, $\mathbf{y}$ has a representing Borel probability measure on $\mathbf{K}$ with a density in $L_\infty(\mathbf{K})$ bounded by c, if and only if $\rho_d < \infty$ for all d.

Each semidefinite program of the hierarchy (2.11), $d \in \mathbb{N}$, has a dual which is also a semidefinite program and which reads:

(2.12)

$$
\begin{array}{rl}
\rho_d^* = \displaystyle\max_{p,q,\sigma_j} & \displaystyle\int_{\mathbf{K}} p(\mathbf{x})\lambda(d\mathbf{x}) + L_{\mathbf{y}}(q) \\
\text{s.t.} & \displaystyle\sum_{(\alpha,k)\in\mathbb{N}^{n+1}_d} (\mathbf{x}^\alpha t^k)^2 - (p(\mathbf{x}) + tq(\mathbf{x})) \;=\; \sigma_0(\mathbf{x},t) + \sum_{j=1}^m \sigma_j(\mathbf{x},t) g_j(\mathbf{x}) \\
& \deg p \leq 2d;\ \deg q \leq 2d-1;\ \sigma_j \in \Sigma[\mathbf{x},t]_{t-v_j},\ j = 0, \ldots, m,
\end{array}
$$

where $v_0 = 0$. In particular, if $\mathbf{y}$ is the sequence of a Borel measure on $\mathbf{K}$ then in (2.12) one may replace $L_{\mathbf{y}}(q)$ with $\displaystyle\int_{\mathbf{K}} q(\mathbf{x})d\mu(\mathbf{x})$.

2.3. **On membership in $L_p(\mathbf{K})$.** An interesting feature of the hierarchy of semidefinite programs (2.11), $d \in \mathbb{N}$, is that it can be used to detect if a given sequence $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, cannot have a representing Borel measure on $\mathbf{K}$ with a density in $L_p(\mathbf{K})$, $p > 1$.

Corollary 2.6. *Let $\mathbf{K} \subset \mathbb{R}^n$ be as in (1.3) and let $\mathbf{y} = (\mathbf{y}_\alpha)$, $\alpha \in \mathbb{N}^n$, be a real sequence with $y_0 = 1$. If the semidefinite program (2.11) with $d \in \mathbb{N}$, has no solution then $\mathbf{y}$ cannot have a representing finite Borel measure on $\mathbf{K}$ with a density in $L_p(\mathbf{K})$, for any $p \geq 2d$.*

Proof. Suppose that $\mathbf{y}$ has a representing measure on $\mathbf{K}$ with a density $f \in L_{2d}(\mathbf{K})$, and hence in $L_k(\mathbf{K})$ for all $k \leq 2d$. Proceeding as in the proof of Theorem 2.5, let ν be the Borel measure on $\mathbf{K} \times \mathbb{R}$ defined in (2.8). Then from (2.9) one obtains

$$z_{\alpha k} \;=\; \int_{\mathbf{K}} \mathbf{x}^\alpha\, f(\mathbf{x})^k\, \lambda(d\mathbf{x}), \quad (\alpha, k) \in \mathbb{N}^{n+1}_{2d},$$

[1] A semidefinite program is a convex optimization problem that can be solved efficiently, i.e., up to arbitrary fixed precision it can be solved in time polynomial in the input size of the problem; see e.g. [1].

which is well-defined since $\mathbf{K}$ is compact (so that $\mathbf{x}^\alpha$ is bounded) and $k \leq 2d$. And so the sequence $\mathbf{z} = (z_{\alpha k})$, $(\alpha, k) \in \mathbb{N}^{n+1}_{2d}$, is a feasible solution of (2.11) with d. $\square$

Notice that again, the detection of absence of a density in $L_p(\mathbf{K})$ is possible with no *à priori* bounding parameter c. But of course, the condition is only sufficient.

Example 1. Let $\mathbf{K} := [0,1]$ and $s \in [0,1]$. Let λ be the Lebesgue measure on $[0,1]$ and let δ_s be the Dirac measure at s. One wants to detect that the Borel probability measure $\mu_a := a\lambda + (1-a)\delta_s$, with $a \in (0,1)$ has no density in $L_\infty(\mathbf{K})$. Then (2.7) reads

$$z_{k0} \,=\, \frac{1}{k+1},\; k = 0,1,\ldots; \quad z_{k1} \,=\, \frac{a}{k+1} + (1-a)s^k,\; k = 0,1,\ldots$$

The set $\mathbf{K}$ is defined by $\{x : g(x) \geq 0\}$ with $x \mapsto g(x) := x(1-x)$. We have tested the conditions $\mathbf{M}_d(\mathbf{z}) \succeq 0$ and $\mathbf{M}_d(g\,\mathbf{z}) \succeq 0$ along with (2.7) where $k \leq 2d$ (for z_{0k}) and $k \leq 2d-1$ (for z_{k1}).

We have considered a Dirac at the points $s = k/10$, $k = 1,\ldots,10$, and with weights $a = 1 - k/10$, $k = 1,\ldots,10$. To solve (2.11) we have used the GloptiPoly software of Henrion et al. [4] dedicated to solving there generalized problem of moments. Results are displayed in Table 1 which should be read as follows:

• A column is parametrized by the number of moments involved in the conditions (2.7). For instance, Column "10" refers to (2.7) with $d = 10/2$, that is, the moment matrix $\mathbf{M}_d(\mathbf{z})$ involves moments z_{ij} with $i + j \leq 10$, i.e., moments up to order 10.

• Each row is indexed by the location of the Dirac δ_s, $s \in [0,1]$ (with $\mu_a = a\lambda + (1-a)\delta_s$). The statement "$1 - a \geq 0.5$" in row "$s = 0.3$" and column "10" means that (2.7) is violated whenever $1 - a \geq 0.5$, i.e., when the weight associated to the Dirac δ_s is larger than 0.5.

One may see that no matter where the point s is located in the interval $[0,1]$, if its weight $1-a$ is above 0.5 then detection of impossibility of a density in $L_\infty([0,1])$ occurs with moments up to order 10. If its weight $1-a$ is only above 0.1 then detection of impossibility occurs with moments up to order 12. So even with a small weight on the Dirac δ_s, detection of impossibility does not require moments of order larger than 12.

Example 2. Still with $\mathbf{K} = [0,1]$, consider now the case where $\mu_a = a\lambda + (1-a)(\delta_s + \delta_{s+0.1})/2$, that is, μ_a is a $(a, 1-a)$ convex combination of the uniform probability distribution on $[0,1]$ with two Dirac measures at the points s and $s + 0.1$ of $[0,1]$, with equal weights. The results displayed in Table 2 are qualitatively very similar to the results in Table 1 for the case of one Dirac.

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$s \setminus$ moments	8	10	12	14
0.0	$1-a \geq 0.3$	$1-a \geq 0.1$	$1-a \geq 0.1$	$1-a \geq 0.1$
0.1	$1-a \geq 1$	$1-a \geq 0.3$	$1-a \geq 0.2$	$1-a \geq 0.2$
0.2	$1-a \geq 1$	$1-a \geq 0.3$	$1-a \geq 0.1$	$1-a \geq 0.1$
0.3	$1-a \geq 1$	$1-a \geq 0.5$	$1-a \geq 0.2$	$1-a \geq 0.1$
0.4	$1-a \geq 1$	$1-a \geq 0.5$	$1-a \geq 0.2$	$1-a \geq 0.1$
0.5	$1-a \geq 1$	$1-a \geq 0.5$	$1-a \geq 0.2$	$1-a \geq 0.1$
0.6	$1-a \geq 1$	$1-a \geq 0.5$	$1-a \geq 0.2$	$1-a \geq 0.1$
0.7	$1-a \geq 1$	$1-a \geq 0.5$	$1-a \geq 0.1$	$1-a \geq 0.1$
0.8	$1-a \geq 1$	$1-a \geq 0.5$	$1-a \geq 0.1$	$1-a \geq 0.1$
0.9	$1-a \geq 1$	$1-a \geq 0.5$	$1-a \neq 0.4, 0.5$	$1-a \geq 0.1$
1.0	$1-a \geq 0.4$	$1-a \geq 0.1$	$1-a \geq 0.1$	$1-a \geq 0.1$

TABLE 1. Moments required for detection of failure; one Dirac

$(s, s+0.1) \setminus$ moments	10	12
(0.1,0.2)	$1-a \geq 0.4$	$1-a \geq 0.2$
(0.2,0.3)	$1-a \geq 0.6$	$1-a \geq 0.2$
(0.3,0.4)	$1-a \geq 0.5$	$1-a \geq 0.2$
(0.4,0.5)	$1-a \geq 0.6$	$1-a \geq 0.1$
(0.5,0.6)	$1-a \geq 0.6$	$1-a \geq 0.2$
(0.6,0.7)	$1-a \geq 0.7$	$1-a \geq 0.2$
(0.7,0.8)	$1-a \geq 0.6$	$1-a \geq 0.3$
(0.8,0.9)	$1-a \geq 0.5$	$1-a \geq 0.2$
(0.9,1.0)	$1-a \geq 0.1$	$1-a \geq 0.1$

TABLE 2. Moments required for detection of failure; two Dirac

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